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Anomalous Generation Numbers in SO(10) and Supersymmetric SO(10) Unification Theories

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Abstract

One of the interesting features in unification models and supersymmetric unification models is that the chiral states of quarks and leptons in a family including a right-handed neutrino can be fitted neatly into a fundamental spinor representation (f.s) of dimension 16 for the SO(10) gauge group. However, it is shown in this paper that such a fundamental spinor representation of SO(10) for Weyl fermions will generate global (non-perturbative) gauge anomalies (of new type) when restricting to the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ gauge subgroup. Such an example is the four SU(2) factors obtained through the reduction of the subgroup $SU(2) \otimes SO(7)$ of SO(10) with the SO(7) to the three SU(2) factors. The branching rule in this case is given by $(f.s) \rightarrow (2-1-2-2) \oplus (2-2-1-2)$ in terms of dimensions. A consistent gauge theory implies the gauge symmetry in a gauge subgroup, and then needs to be well-defined when restricting to the gauge subgroup. Consequently, a consistent SO(10) quantum theory needs to satisfy our selection rule $N_f + N_{mf} = \text{even} \geq 4$, namely the total number of generations with N_f ordinary fermion families and N_{mf} mirror fermion families is even and larger than three, and the three

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generations of chiral fermions in this content can not correspond to a consistent theory. Then we expect that there exist at least one additional fermion family including a right-handed neutrino or at least one family of mirror fermions including a left-handed mirror neutrino if $SO(10)$ unification theory is relevant to our realistic world. The next possibility is to have total six generations that may subject to stringent experimental constraints. Two such more plausible examples are obviously (i) there are three more generations of ordinary fermions including three right-handed neutrinos correspondingly, or (ii) there are three more generations of mirror fermions including three left-handed mirror neutrinos correspondingly. A brief sketch of some physical implications are also given. In the case of mirror fermions, a fundamental aspect is that there will be also V+A currents coupling to the W bosons and the weak interaction is not completely chiral. We also noted the absence of global anomaly in gauge subgroups of the gauge group $SU(5), E_6, E_8, SU(4), SU(6)$, etc. for the relevant representations.

Since the discovery of Yang-Mills theories¹, elementary particle physics has gained great development in the framework of non-abelian gauge theories. With the construction of standard electroweak gauge theory², one of the most interesting ideas has been incorporating the standard model into a grand unified theory³⁻⁴ (GUT) or a supersymmetric grand unified theory⁵⁻⁶. It is now known that although the minimal $SU(5)$ model² is not compatible with proton decay search⁷ and CERN LEP data⁸, the unification may be achieved in either supersymmetric GUT gauge theories or a GUT with a gauge group larger than $SU(5)$ spontaneously broken to the standard gauge group in at least two stages. Such an example⁹ is the $SO(10)$ GUT model which may break¹⁰ first to left-right symmetric $SU(3) \otimes SU(2)_L \otimes SU(2)_R \otimes U_{B-L}$ model at a scale M_X and then to the standard model. While the consideration of gauge hierarchy certainly prefers a supersymmetric $SO(10)$ theory, without evidence of supersymmetry at low energies, non-supersymmetric $SO(10)$ models are still of interest¹⁰. Among many attractive physics features¹¹ of $SO(10)$ models are that they preserve a good prediction for the $\sin^2 \theta_w$, permit neutrinos having small masses through see-saw mechanism, and may give predictions for some parameters in the standard model. Another interesting feature in these models is the structure of the group representation itself, which has all the 16 chiral states of quarks and leptons including the right-handed neutrino that comprises one family fitting neatly into a fundamental spinor representation of dimension 16 for the $SO(10)$ unification gauge group. This is the feature we will focus on in this paper. Especially, we will analyze the global (non-perturbative) effects of this group theoretic structure. Then, the relevant remarks and consequences will be discussed. As a matter of fact, we will see that the $SO(10)$ models with Weyl fermions in a fundamental spinor representation will generate a new type of global (non-perturbative) gauge anomalies¹² which will be clarified soon, and

therefore, the corresponding generating functional is ill-defined, or the quantum theory is inconsistent. This also suggests that the possibilities for this type of global gauge anomalies need to be taken into consideration carefully in general in non-abelian gauge theories with Weyl fermions.

It was shown by Witten¹³ that an $SU(2)$ gauge theory in four dimensions with an odd number of Weyl fermion doublets is mathematically inconsistent due to a global (non-perturbative) gauge anomaly in the theory. With a global gauge anomaly, the generating functional for the quantum theory is ill-defined. Topologically, this is associated with the fact that the homotopy group $\Pi_4(SU(2)) = Z_2$ is non-trivial. In general, for a non-abelian gauge theory in an even dimensions $D=2n$, one still needs to consider the possibility of gauge anomalies if the homotopy group $\Pi_{2n}(G)$ for the gauge group G is non-trivial when the theory is free of local (perturbative) gauge anomalies. Such possible global gauge anomalies have been investigated for $SU(N)$ gauge groups^{14–21}, and systematically and rather generally for arbitrary compact and connected simple gauge groups in generic even dimensions^{15–21}, especially in terms of^{15–21} the James numbers of Steifel manifolds and generalized Dynkin indices. In this paper, we have shown that Weyl fermions in a fundamental spinor representation of $SO(10)$ will generate global gauge anomalies in the theory. Therefore, the conventional $SO(10)$ and supersymmetric $SO(10)$ unification models must be modified in order for the theory to be free of the global gauge anomalies.

The new type global gauge anomalies relevant here are topologically more subtle than the usual ones similar to that first noted by Witten¹³ with the non-trivial homotopy group²² $\Pi_4(G)$ for the gauge group G in four dimensions, since the relevant homotopy $\Pi_4(SO(10)) = 0$ is trivial. Therefore, one might usually expect that there should not be any global gauge anomalies, where note that the $SO(10)$ gauge theories are free of local anomalies in four dimensions in which it

is well known that²³ local gauge anomalies can only arise from Weyl fermions in the complex representations of $SU(N)$ ($N \geq 3$). Even for a simple subgroup H of a gauge group G with $\Pi_{2n}(G) = 0$ being trivial, we showed the following proposition¹⁶. *Proposition:* In arbitrary $D=2n$ dimensions, if the relevant Weyl fermion representation in G free of local gauge anomaly in the strong anomaly cancellation condition $Tr X^{n+1} = 0$ reduces to an irreducible representation ω plus H (simple) singlets (in the generalized convention of allowing negative multiplicities^{15–16}), then there will be no H global gauge anomalies for the representation ω . For the special case with a $SU(2)$ group embedded into a simple gauge group G with $\Pi_4(G) = 0$, it is known that there will be no $SU(2)$ global gauge anomaly if the local anomaly-free condition for the group G is satisfied²⁴.

The new global gauge anomalies we note arises from the restriction of a gauge group G with relevant trivial homotopy group to its gauge subgroups (containing more than one simple ideals with non-trivial forth homotopy group). The example we will focus in this paper is the subgroups of $SO(10)$ due to its crucial importance and relevance to the unification theories. We will now describe our result for the $SO(10)$ gauge theories, and may use the Lie algebras for the discussion of representations. The same notations may be used for the Lie groups and corresponding Lie algebras, no confusion should be caused in our discussion here. The $SO(10)$ group contains a maximal subgroup $SU(2) \otimes SO(7)$ (there is not difference between $SU(2)$ and $SO(3)$ in our consideration since they have the same forth homotopy group). Restricting the $SO(7)$ to the subgroup $SU(2) \otimes SU(2) \otimes SU(2)$, then we obtain a subgroup $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$. For this subgroup, we have the relevant homotopy group

$$\Pi_4(SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \quad (1)$$

The homotopy group topologically classifies the continuous gauge transformations restricted to this subgroup in the compactified spacetime manifold. The

non-trivial topological $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ structures exist when the $SO(10)$ gauge theory is restricted to the subgroup. The possible global gauge anomalies for a $SU(2)$ gauge theory in arbitrary $2n$ dimensions have been determined completely¹⁶ in terms of James numbers of Stiefel manifold (see ref.16 and references therein) and the generalized Dynkin indices^{25–26}. The branching rule²⁷ for a fundamental spinor representation (f.s) of $SO(10)$ in such a reduction is given by

$$(f.s) \rightarrow (1 - 0 - 1 - 1) \oplus (1 - 1 - 0 - 1) \quad (2)$$

in terms of Dynkin labels or

$$(f.s) \rightarrow (2 - 1 - 2 - 2) \oplus (2 - 2 - 1 - 2) \quad (3)$$

in terms of dimensions.

To determine the possible global anomalies, we will first consider an irreducible representation (2-1-2-2) in the branching rule eq.(3), then the overall possibilities can be clarified. For the irreducible representation (2-1-2-2), it is equivalent to consider the possible global anomaly for the group as three $SU(2)$ factors in the irreducible representation (2-2-2). We can embed this into the $SU(8)$ in the fundamental representation \square . Through the reduction of $SU(8)$ to $SU(2) \otimes SU(4)$, and then with the $SU(4)$ to two $SU(2)$ factors, we obtain the three $SU(2)$ factors in the irreducible representation (2-2-2). Since $\Pi_4(SU(8))$ is trivial, the embedding condition is satisfied, in other words, the possible $SU(2) \otimes SU(2) \otimes SU(2)$ global gauge anomalies will appear as the Wess-Zumino term for the $SU(8)$ in the fundamental representation. With the fact $\Pi_5(SU(8)) = Z$, and using exact homotopy sequence it is known^{15–20} that the basic global anomaly coefficient in this case is given by $A = \exp\{i\pi Q_2(\square)\}$. The $Q_2(\square)$ is the second-order Dynkin index for the \square of $SU(8)$ with the possible constraint that the $SU(8)$ gauge theory should be free of local anomaly when

restricting to the three $SU(2)$ factors on the spacetime S^4 as the boundary of a five-dimensional disc D^5 . However, since the three $SU(2)$ factors are automatically free of local gauge anomaly in four dimensions, there is no constraint on the Dynkin index. Therefore^{25–26}, we have $Q_2(\square) = 1$, and the basic global anomaly coefficient in this case is then $A = -1$. We note that^{15–16} in order to obtain the topologically non-trivial $SU(8)$ transformation in the $\Pi_5(SU(8)) = Z$, the corresponding gauge transformation on the D^5 needs to be topologically non-trivial in all the three $SU(2)$ factors when restricting to its boundary S^4 . Only in this case, none of the $SU(2)$ factors may be topologically reduced or equivalently to give a factor 2 from the dimension as a multiplicity for the other $SU(2)$ factors, so that the eight-dimensional (irreducible) embedding is topologically effective. Obviously, we have determined that the basic global anomaly coefficient for the the representation in eq.(3) for the four $SU(2)$ factors is given by

$$A = (-1) \otimes (-1), \quad (4)$$

and the theory with such a restriction has $Z_2 \oplus Z_2$ global gauge anomalies. The consequence is that¹³ the generating functional and the operators invariant under such gauge subgroup cannot be well-defined relative to the relevant large and continuous gauge transformations in the subgroup. The $Z_2 \oplus Z_2$ anomaly may be understood as that when the gauge transformation is topologically non-trivial in three of the $SU(2)$ factors simultaneously but trivial in either the second or the third one in our notation, the fermion measure will change a sign, the quantum theory is then not well-defined¹³. Note that it is necessary for the gauge transformations to be topologically non-trivial in the first and forth $SU(2)$ factors also to generate a global anomaly, since only in this case, they cannot be continuously deformed into identity transformation in these two $SU(2)$ sectors and they will not contribute a factor 2 from each of the dimensions for the two $SU(2)$ factors.

Remark: The two different Z_2 global gauge anomalies for the $Z_2 \oplus Z_2$ arise from the two different irreducible representations in the branching rule eq.(3) and correspond to topologically inequivalent gauge transformations. Therefore, the fact that the $SO(10)$ gauge group with $\Pi_4(SO(10)) = \{0\}$ does not have local gauge anomalies will not contradict to our result. This is because for each of the irreducible representations in eq.(3) (e.g the $(2,1,2,2)$ for the four $SU(2)$ factors), it cannot be embedded into a representation ω of $SO(10)$ such that the ω reduces to the irreducible representation $((2,1,2,2))$ plus singlets upon the reduction. This is also an explicit example showing that the conventional proposition¹⁶ noted by using the Wess-Zumino term argument does not apply generally to the case in which the relevant subgroup has more than one ideals with non-trivial $2n$ -th homotopy group in $D=2n$ dimensions and the representation is not irreducible. This is also why a fundamental representation (f.s) of the $SO(10)$ cannot have $SP(4) \otimes SP(4)$ global anomaly, due to the fact that the (f.s) reduces to the sixteen-dimensional *irreducible* representation $(\square - \square)$ upon the reduction $SO(10) \downarrow SP(4) \otimes SP(4)$, the conventional argument of using Wess-Zumino term for the $SO(10)$ may apply. In this case, the vanishing of $SO(10)$ local anomaly or Wess-Zumino term implies the absence of the relevant $SP(4) \otimes SP(4)$ global anomaly. The problem with embedding a direct sum with more than one irreducible representations of global gauge anomalies is that the anomaly information may not be extracted independently due to the fact that¹⁵ the global gauge anomaly for an irreducible representation free of local gauge anomaly can be at most of Z_2 type. Generally, from this point of view for the global gauge anomalies, the restriction of a gauge theory to a gauge subgroup H may not be the same as embedding the gauge subgroup H into the original gauge group G due to the representation condition needed to analyze the possible global gauge anomalies.

Note that generally gauge symmetry in a gauge group implies the gauge symmetry in its gauge subgroup (see ref.28 for the other studies related to this property), namely a well-defined gauge theory needs to be well-defined when restricting to its gauge subgroups. In quantum theory, if there are gauge anomalies when restricting to a gauge subgroup, then the gauge theory cannot be well-defined. In conclusion, the $SO(10)$ gauge theories with Weyl fermions in a fundamental spinor representation of dimension 16 have global (non-perturbative) gauge anomalies. The $SO(10)$ has two fundamental spinor representations which are complex conjugate to each other. Our results applies to either one of them.

Denote the numbers of two inequivalent fundamental spinor representations as $N(16)$ and $N(\bar{16})$, obviously we need to have

$$N(16) + N(\bar{16}) = \text{even}, \quad (5)$$

in order to cancel out the global gauge anomalies. Consequently, $SO(10)$ unification models with three generations of fermions have global gauge anomalies. We have checked that the adjoint representation of dimensions 45 for the $SO(10)$ will be free of global gauge anomalies for the relevant subgroups. Therefore, our conclusion applies both to the non-supersymmetric $SO(10)$ models and supersymmetric models in which gauginos are in the adjoint representation.

The physics consequences of our result may be of fundamental interest if the $SO(10)$ gauge theories are relevant to the realistic world. Obviously, the $SO(10)$ models and supersymmetric $SO(10)$ unification models need to be modified according to our analysis. In the usual physics convention for the Weyl fermions with the observed three families of leptons and quarks, our selection rule is written as

$$N_f + N_{mf} = \text{even} \geq 4, \quad (6)$$

with the $N_f = N(16)$ and $N_{mf} = N(\bar{16})$ denoting the number of fermion families and the number of mirror fermion families respectively. The above equation

is our main result in this paper. Therefore, we predict that there will be at least one more fermion family or at least one mirror fermion family if an $SO(10)$ unification gauge theory is realistic. Where in the content of $SO(10)$ unification, the fourth generation (or a generation of mirror fermions) also includes a right-handed neutrino (or a left-handed mirror neutrino). Mirror fermions have the same $SU(3) \otimes SU_L(2) \otimes U(1)_Y$ quantum numbers as the ordinary fermions except that they have opposite handedness. Usually²⁹, mirror fermions are considered with three generations. Conventionally, one family of mirror fermions seems not so motivated. However, our result of the global gauge anomalies shows that it is one of the simple ways to cancel the global anomalies. As in the usual discussions, if there exists fourth generation of fermions with V-A weak interaction, then of course, it seems natural to have either no mirror fermions or four families of mirror fermions. From the anomaly-free point of view. The next possibility is either to have three generations of ordinary fermions and three generations of mirror fermions correspondingly as in the usual discussions of mirror fermions, or to have six generations of fermions with three more repetitions of an ordinary fermion family. If there are mirror fermions, one of the most fundamental consequences will then be that the Lorentz structure of the weak interaction will no longer be chiral with only V-A currents coupling to the W gauge bosons, there will be also V+A piece which though may be very small relevant to the current experimental observation²⁹. There has been analysis²⁹ about the charged and neutral current data suggesting that the possible V+A impurity in the weak amplitudes is typically less than about 10%.

We will now give a brief sketch of some other related physics issues, for details see the relevant references. In the content of the electroweak theory, either an additional generation of fermions or a generation (three generations) of mirror fermions obtain their masses through the electroweak symmetry breaking at the

order of about $O(300\text{Gev})$, this will give effects on low energy physics and also subject to both theoretical and experimental constraints. The LEP data set a lower bound for their masses denoted by M_F at about $M_F \geq m_z/2$, namely about half of the Z boson mass. The partial wave unitarity³⁰ at high energies shows that the masses above about $O(600\text{Gev}/\sqrt{N_{DQ}})$ and $O(1\text{TeV}/\sqrt{N_{DL}})$ for quarks (or mirror quarks) and leptons (mirror leptons) will signal the breakdown of the perturbation theory, where N_{DQ} denotes the total number of nearly degenerate weak-isospin doublets for quarks and mirror quarks, and similarly with N_{DL} for leptons and mirror leptons. There may be stringent constraint on the masses, mass splittings in a weak-isospin doublet for possible new fermions due to the bound on the correction $\delta\rho$ for the parameter^{31,32} $\rho = m_w^2/m_z^2 \cos^2 \theta_w$ from its tree level value in the minimal standard model, as well as for the other precision electroweak parameters^{31,33}. It is known that the radiative corrections³¹ in perturbation theory can play an active role in this. There are also recent discussions that³² the possible bound states formed by the exchange of Higgs bosons in the presence of additional heavy fermions may give non-perturbative contribution $\delta\rho < 0$ and cancel the perturbative correction within the current experimental error in the nearly degenerate case, and therefore can relax the constraints for the masses and mass splittings due to the ρ parameter.

An additional family of fermions or mirror fermions may also be constrained by that Yukawa couplings should remain small during the evolution in the perturbative region (about $\alpha_{Y_{uk}} = \lambda_{Y_{uk}}^2/4\pi \leq 1$), otherwise its running may induce Landau poles in the one-loop approximation. The presence of these singularities at some scale signals the breakdown of perturbation theory and the probable triviality of the continuum limit. Related to the running of Yukawa couplings and infrared fixed-point solution³⁴ to the renormalization group equations, there has been discussions^{34,35} in supersymmetric unification models with

Yukawa coupling unification (e.g. $\lambda_\tau = \lambda_b$ at the unification scale) that only small regions in the $m_t - \tan \beta$ ($\tan \beta = v_{up}/v_{down}$) plane may be allowed (e.g. about $1 \leq \tan \beta \leq 1.5$ or $\tan \beta \geq 40 \pm 10$ with $m_t \leq 175 \text{ GeV}$). The value of $\tan \beta$ can typically effect^{34–35} the flavor changing neutral currents in processes like $b \rightarrow s\gamma$ and $B\bar{B}$ mixing and to the proton decay³⁶. It also constraints on the Higgs boson masses³⁷ which may be relevant to the LEP II. If there are additional fermions or mirror fermions, one may expect that their presence will also have these typical effects, relevant discussions with Yukawa coupling unification at the unification scale then may need to incorporate them. At least, the possibility of one additional generation of mirror fermions from our motivation in terms of global gauge anomalies sounds quite new. It has been argued³⁸ in the conventional mirror fermion models (with three generations of mirror fermions corresponding to ordinary observed generations of quarks and leptons) that mirror doublets should always be assumed degenerate in masses (without considering the non-perturbative effects in ref. 32) in order to reproduce the precision LEP data, and the possible Higgs masses may be in rather restricted regions. We note that mirror fermions of at least three generations usually appear in the particle spectrum of many theories other than some superstring models with family unification, such as those with extended supersymmetry ($N \geq 2$) imposed on a gauge theory, in the Kaluza-Klein theories³⁹, and some composite models⁴⁰. However, according to our analysis of global anomalies in SO(10) unification gauge theories, one of the interesting models is to have only one generation of mirror fermions besides the three generations of ordinary fermions. In general, one may expect that^{29,38} mirror fermions need to mix with the ordinary fermions in order to avoid stable mirror fermions although the mixing may be small. If there exists only one generation of mirror fermions, fundamentally, it is unnatural to assume that it corresponds to a particular family of ordi-

nary fermions. Therefore, this generation of mirror fermions will mix with all the three generations of ordinary fermions, and this then will induce the flavor mixing between the three generations of ordinary fermions also, this seems to provide another origin for the possible flavor mixing and possible CP violations. Moreover, as it is known that the lifetime of heavy neutrinos may subject to cosmological constraint⁴¹ (a suggestion is about $\tau < 10^3 yr (1kev/m_v)^2$) since it may be strongly believed that the age of the universe is greater than 10^{10} years. Our selection rule may have consequences on the structure of the universe.

According to our rule in eq.(6) for SO(10) unification gauge theories, since the total number N_t of generations for ordinary fermions and mirror fermions needs to be $N_t = even \geq 4$, the models with family unification beyond SO(10) theories with the other possibilities we discussed above other than the case $N_f = N_{mf} = 3$ are then better motivated. The models with only $N_t = N_f = 3$ or $N_t = N_f + N_{mf} = odd$ when the gauge symmetry is broken down to the SO(10) then may not be consistent. Our rule for the generation numbers in eq.(6) may be of fundamental interest and importance. We will conclude our paper with the following related discussions.

(i) We have also noted that the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ is the only possible subgroup up to isomorphism having $Z_2 \oplus Z_2$ global gauge anomalies. Another example of the reduction is the four SU(2) factors obtained through the reduction of the subgroup $SU(2) \otimes SU(2) \otimes SO(6)$ with the SO(6) to two SU(2) factors. But there may be possibly $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ anomaly in the four SU(2) factors obtained through the reduction of the subgroup $SP(4) \otimes SP(4)$ of SO(10) with each SP(4) to two SU(2) factors, since this reduction leads to a different branching rule. Moreover, the SO(10) group in low-dimensional representations of dimensions 45, 54, 120 etc. will not have global gauge anomalies. Especially, we emphasize that since in the supersymmetric SO(10) models, the gauginos are

in the 45-dimensional adjoint representation, our selection rule eq.(6) applies both to non-supersymmetric $SO(10)$ and supersymmetric $SO(10)$ unification theories.

(ii) For Weyl fermions in the $(\square - \square)$ for the $SU(2) \otimes SU(2)$, it can be embedded into the four-dimensional \square of $SU(4)$ with $\Pi_5(SU(4)) = Z$. One can easily see that there may be possibly a Z_2 global gauge anomaly in this case. This fact is relevant to the possible $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ global anomaly in (i) for a (f.s) of $SO(10)$. By using this fact, one can also check that the vector representation of dimension 10 for the $SO(10)$ may have global gauge anomalies when restricting to certain subgroups as product of $SU(2)$ factors through many reductions. But they all cancel out if the total number of the vector representations are even. Some examples are given below. (1) $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ global anomalies when restricting to the $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$ through the reduction of subgroup $SP(4) \otimes SP(4)$ of $SO(10)$; (2) $Z_2 \oplus Z_2$ global gauge anomalies when restricting to a $SU(2) \otimes SU(2)$ gauge subgroup through the following reductions: (2a) The $SU(2) \otimes SU(2)$ from the reduction of either one of the $SP(4)$ factors in (1); (2b) the $SU(2) \otimes SU(2) \otimes SO(6)$ ($SO(6) \cong SU(4)$) subgroup of $SO(10)$ to the two $SU(2)$ factors; (2c) by the reduction of the subgroup $SU(2) \otimes SO(7)$ of $SO(10)$ to $SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$, the relevant two $SU(2)$ factors both are in the fundamental representation in one of the irreducible representations (2-2-1) in the branching rule for the reduction of the $SO(7)$ to $SU(2) \otimes SU(2) \otimes SU(2)$. Obviously, the global gauge anomalies in a fundamental representation cannot be canceled by adding more vector representations and vice versa. It is a typical feature that the branching rules become much more involved when the dimension goes higher, if there were other possible anomalies they would be very dependent on the reduction procedure, global anomalies from representations of different dimensions may not cancel each other. At least, we checked up to

dimensions at least several hundred, no other higher irreducible representations can have the same possibilities for the global anomalies as that of a fundamental spinor representation. Our selection rule for the generation numbers in eq.(6) is realistically general.

(iii) For superstring theory with gauge group $E_8 \times E'_8$, a compactification of the heterotic string is $E_6 \times E'_6$, N=1 supersymmetric Yang-Mills theory in four dimensions⁴². In the E_6 sector, the left-handed Weyl fermions are in the real representation $\{78\} \oplus 3 \times \{27\} \oplus 3 \times \{\bar{27}\} \oplus 8 \times \{1\}$ of the E_6 in terms of the dimensions. In four dimensions, the E_6 is a local anomaly-free group with $\Pi_4(E_6) = 0$ being trivial. We can show that this representation will not have a global anomaly when restricting to a subgroup with non-trivial fourth homotopy group. In this case of E_6 upon the reduction to $SO(10)$, it can also be seen more obviously by our analysis. Upon the reduction $E_6 \downarrow SO(10)$, $78 \rightarrow 45 \oplus 16 \oplus \bar{16} \oplus 1$, and $27 \rightarrow 16 \oplus 10 \oplus 1$ and correspondingly for the $\bar{27}$. After the decomposition, there is a $\bar{16}$ for each 16 and the 10 also appear in pairs. Therefore, there will be no global gauge anomalies upon reduction to $SO(10)$. One can also see explicitly that¹² the theory have no global gauge anomaly for the other possible gauge subgroups with non-trivial fourth homotopy group. Therefore, the relevant heterotic string theory is free of both local and global gauge anomalies. An E_6 unification gauge theory with even total number for 27 and its complex conjugate $\bar{27}$ is free of global gauge anomaly.

(iv) For the $SU(5)$ and supersymmetric $SU(5)$ theories, we have shown that the relevant Weyl fermion representations (e.g. $\bar{5} \oplus 10$) free of local gauge anomaly are also free of global gauge anomaly for the gauge subgroups (such as $SP(4)$ and $SU(2) \times SU(2)$ etc.), as well as^{24,16} $SU(2)$ with non-trivial fourth homotopy group. Moreover, for the relevant representations of E_8 , $SU(4)$ (for example $8\{4\} \oplus \bar{10}$), and $SU(6)$ gauge groups free of local gauge anomaly, it can

be seen that¹² they are free of global gauge anomaly for the other subgroups with non-trivial relevant homotopy group. The central result of the paper is our rule of selecting the even total generation numbers for ordinary fermions and mirror fermions. Explicit details of the other discussions above are too involved to be given here. Our analysis also suggests that a new direction of research is to consider the possible gauge anomalies in the subgroups larger than $SU(2)$, especially for those containing more than one simple ideals with non-trivial forth homotopy groups. A general and useful result is that¹⁶ in $D=4$ dimensions (mod 8), $SU(2)$ has Z_2 -type global gauge anomalies only if an irreducible representation has the spin of the form $J=(1+4k)/2=1/2, 5/2, 9/2, \dots$

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